Poisson-Summationsformel

Sind f und $\hat{f}$ stetig und quadratintegrierbar, so gilt

$$\sum_{j\in\mathbb{Z}} f(j) = \sum_{\ell\in\mathbb{Z}} \hat{f}(2\pi\ell)\,.$$

Beweis:

formale Argumentation:
Definition der inversen Fourier-Transformation und Umformung $\leadsto$

$$\begin{aligned}\sum_{j\in\mathbb{Z}} f(j) &= \frac{1}{2\pi}\sum_j \int_{-\infty}^{\infty} \hat{f}(y)e^{\mathrm{i}jy}\,dy = \frac{1}{2\pi}\sum_j\sum_\ell \int_{2\pi\ell-\pi}^{2\pi\ell+\pi} \hat{f}(y)e^{\mathrm{i}jy}\,dy \\ &= \sum_j \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_\ell \hat{f}(y+2\pi\ell)\right]e^{\mathrm{i}jy}\,dy\end{aligned}$$

$[\ldots] = g(y)$: 2π-periodisch
$\implies$ $\sum_j f(j)$: Summe s der Fourier-Koeffizienten c_{-j} von g, d.h.

$$s = \sum_j c_j e^{\mathrm{i}jy}\Big|_{y=0} = g(0) = \sum_\ell \hat{f}(2\pi\ell)$$

Beispiel:

Hutfunktion

$$g(x) = \begin{cases} 0, & x < -1 \\ x+1, & -1 \le x < 0 \\ 1-x, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases}, \quad \hat{g}(y) = \operatorname{sinc}^2(y/2) = \frac{\sin^2(y/2)}{(y/2)^2}$$

Poisson-Summationsformel mit

$$f(x) = \exp(\mathrm{i}ax)g(x) \quad \stackrel{\mathcal{F}}{\longmapsto} \quad \hat{g}(y-a) = \hat{f}(y)$$

$\Longrightarrow$

$$1 = \sum_{j\in\mathbb{Z}} g(j)\mathrm{e}^{\mathrm{i}ja} = \sum_{\ell\in\mathbb{Z}} \frac{\sin^2(\pi\ell - a/2)}{(\pi\ell - a/2)^2} = \frac{1}{\pi^2}\sum_{\ell\in\mathbb{Z}} \frac{\sin^2(a/2)}{(\ell - a/(2\pi))^2}$$

denn $g(j) = 0$ für $j \neq 0$ und $\sin^2 x$ ist π-periodisch nach Umformung

$$\frac{\pi^2}{\sin^2(a/2)} = \sum_{\ell\in\mathbb{Z}} \frac{1}{(\ell - a/(2\pi))^2}$$

$a = \pi \quad \rightsquigarrow$

$$\pi^2 = \sum_{\ell\in\mathbb{Z}} \frac{1}{(\ell - 1/2)^2} = \sum_{\ell\in\mathbb{Z}} \frac{4}{(2\ell - 1)^2}$$

$\Leftrightarrow$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$